Upper and lower critical dimension, Goldstone modes, Mermin-Wagner theorem and basics of KT-transition.

Ginzburg criteria: In mean-field theory we ignore correlation of fluctuations.

For example $\langle\sigma_i\sigma_j\rangle \simeq \langle\sigma_i\rangle\langle\sigma_j\rangle$ equivalently $\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \simeq 0$

In Landau theory, thermodynamic averages are determined from saddle point of partion sum

$$Z = \int \mathcal{D}[\phi]\, e^{-F[\phi]} \sim e^{-F[\phi^*]} \quad \text{such that } \langle\phi\rangle = \phi^*$$

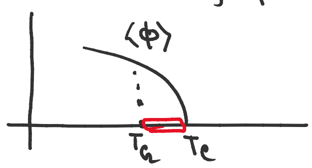
Therefore we ignore fluctuations, eg,

$$\langle(\phi(\vec r)-\phi^*)(\phi(\vec r')-\phi^*)\rangle = \langle\phi(\vec r)\phi(\vec r')\rangle - (\phi^*)^2 \simeq 0.$$

[In Curie-Wiess meanfield we replace $\sum_{\langle ij\rangle}\sigma_i\sigma_j \longrightarrow \sigma_i \frac{1}{N}\langle\sum_j\sigma_j\rangle = \sigma_i h_{mf}$
Because of large N, fluctuations are suppressed]

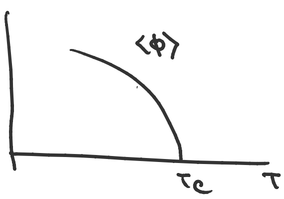
On the other hand, near critical point fluctuations are large, so we naively expect mean field theory to fail. Yet in many real life examples mean field theory correctly captures the critical behavior. For example, low T_c superconducting transition.

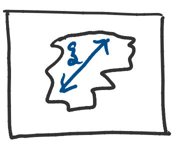
Ginzburg criteria gives an estimate for when MFT breaks down, specially in what range of T_c.



The estimation: There are different approaches. For example, Kardar approaches from error estimate in Saddle point of Landau theory. They all are equivalent. We take a simple handwaving argument.

Consider an ordered phase $T<T_c$





If $T \ll T_c$: correlation length ξ is small. Then MFT is a good approximation.

As we go closer to T_c, ξ grows. Then the error we make can be estimated by

$$E_g = \frac{\sum_i^{\xi}\sum_j^{\xi}\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle}{\left(\sum_i^{\xi}\langle\sigma_i\rangle\right)\left(\sum_j^{\xi}\langle\sigma_j\rangle\right)} = \frac{\int^{\xi} d\vec r\int^{\xi} d\vec r'\,\langle\phi(\vec r)\phi(\vec r')\rangle - \langle\phi(\vec r)\rangle\langle\phi(\vec r')\rangle}{\left(\int^{\xi} d\vec r\,\langle\phi(\vec r)\rangle\right)^2}$$

$$\boxed{\text{If } E_g \ll 1 \quad \text{then MFT is good.}}$$

Let's see what can we say about it:

$$\int^{\xi} d\bar{r} \langle \phi(\bar{r}) \rangle \simeq \xi^d \, \bar{\Phi} \qquad \text{bulk order parameter.}$$

$$\int^{\xi} d\bar{r} \int^{\xi} d\bar{r}' \langle \phi(\bar{r}) \phi(\bar{r}') \rangle - \langle \phi(\bar{r}) \rangle \langle \phi(\bar{r}') \rangle \simeq \int^{\xi} d\bar{r} \int^{\xi} d\bar{r}' \, G(\bar{r} - \bar{r}')$$

$$\simeq \xi^d \int^{\xi} d\bar{r} \, G(\bar{r})$$

$$\simeq \xi^d \chi_T \quad \leftarrow \text{Static susceptibility. (using fluctuation dissipation relation)}$$

This gives

$$\boxed{E_g \simeq \frac{\xi^d \chi_T}{\xi^{2d} \bar{\Phi}^2}}$$

Near a critical point

$$\chi_T \sim \frac{1}{|T - T_c|^{\gamma}}, \qquad \xi \sim \frac{1}{|T - T_c|^{\nu}}, \qquad \bar{\Phi} \sim |T - T_c|^{\beta}$$

Then

$$\boxed{E_g \simeq |T - T_c|^{-\gamma + d\nu - 2\beta} \ll 1} \quad \text{then MFT is good.}$$

Cases: ① the criteria gives an estimate how close to T_c, MFT is good. In many real world examples this window ΔT_g is much smaller than experimental resolution. For example in low temp super conductors $\Delta T_g \sim 10^{-18}$ [Kardar, vol 2], and that's why BCS theory is good.

② For $-\gamma + d\nu - 2\beta > 0 \Rightarrow d > d_u = \dfrac{\gamma + 2\beta}{\nu}$ MFT is always good.

Here d_u is the upper critical dimension.

For Ising-type systems (fluid, magnets), using the known critical exponents

$$d_u = 4.$$

$\Rightarrow$ for $d > d_u$: MFT estimate of critical behavior is good.

for $d = d_u$: Power-law dependences has logrithmic corrections.

For further readings: see Kardar vol 2, Goldenfeld book, and book of Kadanoff.

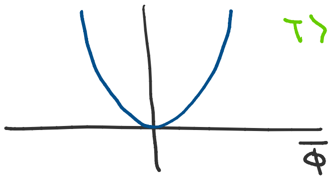
A short summary of Landau theory (only essentials. More details from termpaper presentation on Friday)

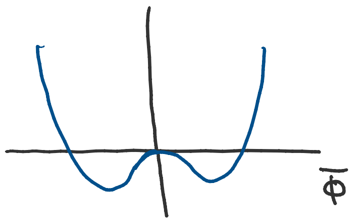
Basic Idea: thermodynamic average of order parameter $\phi(\vec{r})$ is given by minimum of phenomenological Landau-Ginzburg energy functional

For example, in Ising model, $\phi(\vec{r}) \cong m(\vec{r})$ (a scalar field)

$$\mathcal{L}[\phi] = \int d^d\vec{r} \left\{ -h(\vec{r})\,\phi(\vec{r}) + a \cdot \frac{T-T_c}{T_c} \cdot \phi(\vec{r})^2 + \frac{1}{2} b (\phi(\vec{r}))^4 + \frac{R}{2} (\vec{\nabla}\phi(\vec{r}))^2 \right\}$$

A second order transition





Remark: One way to motivate Landau theory is by coarse-graining.

$$Z = \sum_{\{\sigma\}} e^{-H} \longrightarrow \int \mathcal{D}[\phi]\, e^{-\mathcal{L}[\phi]}$$

On a coarse grain scale $1/\Lambda$ such that

$$a \ll \frac{1}{\Lambda} \ll \xi$$

microscopic scale

correlation length of ϕ fields.

How to calculate spatial correlations within Landau-Ginzburg theory.

Primer: basics of fourier modes

$$\phi(\vec{r}) = \frac{1}{L^d} \sum_{\vec{k}}^{\Lambda} \hat{\phi}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

L^d periodic

$$\vec{k} = \left\{ \frac{2\pi}{L} n_1, \ldots, \frac{2\pi}{L} n_d \right\}$$

Invers transformation

$$\hat{\phi}(\vec{k}) = \int^L d\vec{r}\; \phi(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

Basic identity

$$\delta(\bar{r}) = \frac{1}{L^d} \sum_{\bar{k}}^{\wedge} e^{i\bar{k}\cdot\bar{r}} \qquad \text{and} \qquad L^d \, \delta_{\bar{k},\bar{k}'} = \int d\bar{r} \; e^{-i(\bar{k}-\bar{k}')\cdot\bar{r}}$$

Moreover,

$$\int \mathcal{D}[\phi] \longrightarrow \int \prod_{\bar{k}} d\hat{\phi}(\bar{k})$$

A useful relation : fluctuation dissipation relation. (use of functional derivative)

$$\langle \phi(\bar{r}) \rangle = \frac{1}{Z[h]} \int \mathcal{D}[\phi] \cdot \phi(\bar{r}) \cdot e^{-\mathcal{L}[\phi, h]}$$

$$= \frac{\delta}{\delta h(\bar{r})} \log Z[h]$$

$$\Rightarrow \chi(\bar{r}, \bar{r}') = \frac{\delta \langle \phi(\bar{r}) \rangle}{\delta h(\bar{r}')} = \frac{\delta^2}{\delta h(\bar{r})\, \delta h(\bar{r}')} \log Z[h]$$

$$G(\bar{r}, \bar{r}') = \langle \phi(\bar{r}) \phi(\bar{r}') \rangle - \langle \phi(\bar{r}) \rangle \langle \phi(\bar{r}') \rangle$$

$$= \frac{\delta^2}{\delta h(\bar{r})\, \delta h(\bar{r}')} \log Z[h]$$

gives

$$\chi(\bar{r}, \bar{r}') = G(\bar{r}, \bar{r}')$$

$$\Big\downarrow \text{ translation invariance}$$

$$\chi(\bar{r} - \bar{r}') = G(\bar{r} - \bar{r}')$$

$$\Big\downarrow \text{ Fourier transform}$$

$$\boxed{\hat{\chi}(0) = \hat{G}(0) = \int d\bar{r} \; G(\bar{r})} \Longleftarrow \boxed{\hat{\chi}(\bar{k}) = \hat{G}(\bar{k})}$$

Static susceptibility.

Now, calculation of correlation in Landau-Ginzburg theory :

Step 1 : Find $\chi(\bar{r})$ Step 2 : use $G(\bar{r}) = \chi(\bar{r})$

Step 1 : in Landau theory, $\langle \phi(\bar{r}) \rangle = \phi^*(\bar{r})$ is given by saddle point of $\mathcal{L}$.

$$\frac{\delta \mathcal{L}}{\delta \phi^*(\bar{r})} = 0 \Rightarrow -h(\bar{r}) + 2a\underset{\uparrow \, \frac{T - T_c}{T_c}}{t} \, \phi^*(\bar{r}) + 2b[\phi^*(\bar{r})]^3 - R \nabla^2 \phi^*(\bar{r}) = 0$$

Then, taking one more derivative $\frac{\delta}{\delta h(\bar{r}')}$ and realizing $\chi(\bar{r} - \bar{r}') = \frac{\delta \phi^*(\bar{r})}{\delta h(\bar{r}')}$

we get

we get

$$-\delta(\bar{r}-\bar{r}') + 2at\,\chi(\bar{r}-\bar{r}') + 6b\,\phi^{*2}(\bar{r})\cdot\chi(\bar{r}-\bar{r}') - R\,\nabla_r^2\chi(\bar{r}-\bar{r}') = 0$$

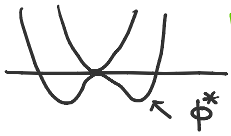
$$\Rightarrow \left[-R\nabla_r^2 + 2at + 6b\,\phi^{*2}(\bar{r})\right]\underbrace{\chi(\bar{r}-\bar{r}')}_{\Updownarrow\, G(\bar{r}-\bar{r}')} = \delta(\bar{r}-\bar{r}')$$

Therefore,

$$\left(-\nabla^2 + m^2\right) G(\bar{r}-\bar{r}') = \frac{1}{R}\,\delta(\bar{r}-\bar{r}')$$

here $m^2 = \frac{1}{R}\left\{2at + 6b\,\phi^{*2}\right\}$ is given by saddle points of $\mathcal{L}$

$$= \begin{cases} \sqrt{-\frac{4a}{R}t} & \text{for } t<0 \\ \sqrt{\frac{2at}{R}} & \text{for } t>0 \end{cases}$$



$\boxed{\text{for } h\to 0 \text{ case}}$

Remark: Note, $G(\bar{r}-\bar{r}')$ is the Greens function of the theory.

Fourier mode: (Important. remember this)

$$\left(k^2+m^2\right)\hat{G}(\bar{k}) = \frac{1}{R} \Rightarrow \boxed{\hat{G}(\bar{k}) = \frac{1/R}{k^2+m^2}}$$

What does this form tell us?

Inverting the transformation

$$G(\bar{r}) = \frac{1}{L^d}\hat{\sum_{\bar{k}}}\frac{1/R}{k^2+m^2}e^{i\bar{k}\cdot\bar{r}}$$

$$\longrightarrow \frac{1}{(2\pi)^d}\int_{1/L}^{\Lambda} d\bar{k}\,\frac{1/R}{k^2+m^2}e^{i\bar{k}\cdot\bar{r}}$$

$$= \frac{1}{R(2\pi)^d}\cdot\frac{1}{r^{d-2}}\int_{r/L}^{\Lambda r} d\bar{q}\,\frac{e^{iqr\cos\theta}}{q^2+(mr)^2}$$

$$\sim \frac{e^{-mr}}{r^{d-2}} \quad \text{for } d>2 \text{ and } r \gg 1/\Lambda .$$

Remark: (i) this form for correlations $G(r) \sim \frac{1}{r^a}\cdot e^{-mr}$ is Ornstein-Zernike form.

(2) The correlation length $\xi = 1/m$. In Landau theory

$$\xi = \frac{1}{m} \sim \frac{1}{\sqrt{|T-T_c|}} \Rightarrow \text{critical exponent } \nu = \frac{1}{2}.$$

(3) at critical point $(T=T_c)$, mass $\boxed{m=0} \Rightarrow \xi \to \infty$, and there is power-law tail for correlations.

$$\boxed{G(r) \sim \frac{1}{r^{d-2}}} \quad \text{for } d>2 \text{ in MFT.}$$

What we learned above will be useful now. [ch 11 of Goldenfeld book]

Goldstone mode, Mermin-Wagner theorem, lower-critical dimension.

Landau theory for O(n) model:

$$H = -J \sum_{\langle ij \rangle} \bar{\sigma}_i \cdot \bar{\sigma}_j \qquad \text{with } \bar{\sigma} \equiv \text{an } n\text{-component unit vector.}$$

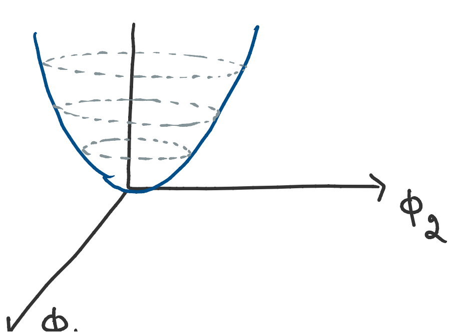
Corresponding Landau free energy functional

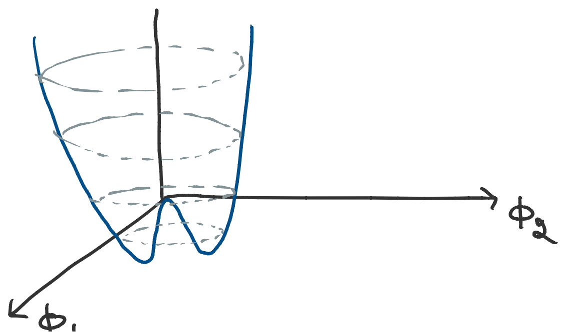
$$\mathcal{L} = \int d^d\bar{r} \left\{ -\bar{h}(\bar{r}) \cdot \bar{\Phi}(\bar{r}) + a \cdot t \cdot (\bar{\Phi} \cdot \bar{\Phi}) + \frac{1}{2} b (\bar{\Phi} \cdot \bar{\Phi})^2 + \frac{R}{2} \sum_{i=1}^{d} (\partial_{x_i} \bar{\Phi}) \cdot (\partial_{x_i} \bar{\Phi}) \right\}$$

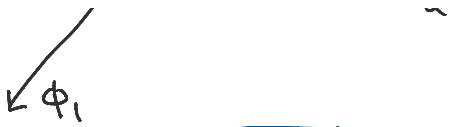
$$= \mathcal{L}_0[\bar{\Phi}, \bar{h}] + \mathcal{L}_{II}[\bar{\Phi}]$$

* This functional preserves the O(n) rotational symmetry.
* Be careful that there are two vector spaces. (1) spatial directions (d dim)
  (2) Space of $\bar{\Phi}$ vectors. (n dim)
* $\bar{h}$ is a vector field on ϕ-space.

Let's first consider only the homogeneous part $\mathcal{L}_0$:







$\boxed{\text{For } t > 0.}$

$\langle \bar{\Phi} \rangle = 0$

$\boxed{\text{For } t < 0.}$

$\langle \bar{\Phi} \rangle = \phi_0 \cdot \hat{e}_1$

a spontaneously broken phase (say in $\hat{e}_1$ direction)

Q. The question is does this spontaneously broken phase survive?

<u>Let's see this qualitatively:</u> (see Goldenfeld)

(1) Looking at the Mexican hat shape we could expect that <u>transverse</u> fluctuations would be large and destroy the spontaneous broken phase. It requires almost no energy for such fluctuations.

This can be seen quantitatively:

longitudinal correlation: $\langle \phi_1(\bar{r})\phi_1(\bar{r}') \rangle - \langle \phi_1(\bar{r}) \rangle \langle \phi_1(\bar{r}') \rangle = G_{11}(\bar{r}-\bar{r}) = \chi_{11}(\bar{r}-\bar{r}')$

transverse correlation: any other $(n-1)$ directions $= G_\perp(\bar{r}-\bar{r}') = \chi_\perp(\bar{r}-\bar{r}')$

A simple algebra (see Goldenfeld) shows

(using only $\mathcal{L}_0$)

$$\hat{\chi}_\perp(0) = \lim_{h \to 0} \frac{\phi_0}{h} = \infty \quad \text{in a symmetry broken phase } (\phi_0 \neq 0).$$

static transverse susceptibility

$$\Rightarrow \hat{G}_\perp(0) = \hat{\chi}_\perp(0) = \infty$$

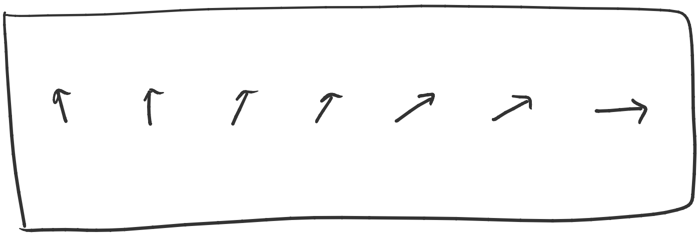
$$\Rightarrow \int d\bar{r} \, \langle \phi_\alpha(\bar{r})\phi_\alpha(0) \rangle - \langle \phi_\alpha(\bar{r}) \rangle \langle \phi_\alpha(0) \rangle \to \infty$$

This means there is no energy cost for transverse fluctuations.

Q: does this conclusion hold in presence of the $\mathcal{L}_{II}$ term?

$$\mathcal{L}_{II} = \frac{R}{2} \int d\bar{r} \sum_{\alpha=1}^{d} (\partial_{x_\alpha} \bar{\Phi}) \cdot (\partial_{x_\alpha} \bar{\Phi})$$

Clearly this term costs energy for transverse fluctuations. It is $\sim (\nabla \phi)^2$.



We see that longer the wavelength of spatial variation, lower the cost.

In a spontaneously symmetry broken phase of $O(n)$ model, there are excitations with modes of arbitrary low energy. There are such modes for each $n-1$ directions. These are called <u>Goldstone modes.</u>

This statement can be made more quantitative by explicitely calculating correlation for the Landau functional $\mathcal{L}_0 + \mathcal{L}_{II}$. The analysis is similar to what we did for scaler ϕ field. (See Goldenfeld ch 11.1.2)

$$\hat{G}_{\parallel}(\bar{k}) = \frac{*}{k^2 + m^2} \quad \text{and} \quad \boxed{\hat{G}_{\perp}(\bar{k}) = \frac{*}{k^2}}$$

$\uparrow$ 4a|t|

These are Goldstone modes for $k \to 0$.

$\Rightarrow$ transverse correlations $G_{\perp}(r) \sim 1/r^{d-2}$ for $d > 2$.

$\Rightarrow G_{\parallel}(r) \sim e^{-mr}$

exponential decay of correlation.

Remark: (1) These modes are a consequence of $O(n)$ symmetry of $\mathcal{L}$, and works with general continuous symmetries. Goldstone's theorem predics such modes in a general state where a continuous symmetry is spontaneously broken. (does not apply in presence of Guage fields).

Ref. Book of DJ Amit, on field theory

(2) Noting $\hat{G}_{\perp}(k)$ are propagators, in QFT, they are similar to bare propagator of massless Bosons.

(3) Real-life examples of Goldstone modes are phonons, spin waves etc.

The big question is, now with full $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{II}$, does the spontaneously symmetry broken phase survive in presenee of these Goldstone modes?

The answer lies in the formula for transverse fluctuations (from the Goldstone modes)

$$G_{\perp}(\bar{r}) = \int_{1/L}^{\Lambda} \frac{d\bar{k}}{(2\pi)^d} \cdot \frac{e^{i\bar{k}\cdot\bar{r}}}{(k)^2}$$

$$\Rightarrow \langle \phi_{\perp}^2(0) \rangle - \underbrace{\langle \phi_{\perp}(0) \rangle^2}_{0} = \int_{1/L}^{\Lambda} \frac{d\bar{k}}{(2\pi)^d} \cdot \frac{1}{k^2} \sim \begin{cases} L - 1/\Lambda & \text{for } d = 1 \\ \log(L\Lambda) & \text{for } d = 2 \\ \Lambda^{d-2} - \frac{1}{L^{d-2}} & \text{for } d \geq 3 \end{cases}$$

We see that transverse fluctuations are finit in the $L \to \infty$ limit for $d \geq 3$. Therefore it is reasonable that symmetry broken state survive.

For $d \leq 2$, fluctuations are infinite $\Rightarrow$ symmetry broken phase donot survive the Goldstone modes.

This is essentially the statement of Mermin-wagner theorem.

$\boxed{\text{No spontaneously broken phase for } d \leq 2 \text{ in } O(n) \text{ model for } n \geq 2. \quad \Rightarrow \text{lower critical dimension} = 2}$

Remarks: Following the correspondance between d-dim classical stat mech model to a d-1 dimensional quantum model, one expects there will be no spontaneous symmetry breaking in 1d quantum system.

Original MW statement is about quantum spin chain.

Mermin-wagner theorem: "at finite temperatures, the quantum spin-S heisenberg model with isotropic and finite-range exchange interactions on one or two dimensional lattices can neither be ferro nor antiferromagnetic."

Ref: Scholarpedia article by Wagner.

The MW conclusion is powerful and applies with much generality.
A well-known example is lack of 2d crystal, ie no spontaneous translation symmetry breaking. A famous argument follows the similar line of reasoning as above. It starts with a possible 2d regular arrangement (crystal) and makes harmonic approximation for deviations around this positions. (harmonic solids)

Then shows that these fluctuations diverge in 2d

$$\langle (u(\vec{r}) - u(0))^2 \rangle \sim \log r$$

Therefor, the crystal structure is not stable against any thermal fluctuations, at any non-zero temp.
[Work of Mermin and later by Halperin, Nelson, thouless ...]

Q: In 2-d there is an amorphous solid phase where orientational order parameter $\psi_6(\vec{r})$ has true long-range order. How is this commensurate with MW theorem?